Exercise 11

Use power series to solve the differential equation.

$$y'' + x^2y' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x^2 and x inside the respective summands.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Because of n in the summand, the second series can start from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Make the substitution $n = k + 3$ in the first series, the substitution $n = k$ in the second series, and the substitution $n = k$ in the third series.

$$\sum_{k+3=2}^{\infty} (k+3)[(k+3)-1] a_{k+3} x^{(k+3)-2} + \sum_{k=0}^{\infty} k a_k x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = 0$$

Simplify the first sum.

$$\sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} + \sum_{k=0}^{\infty} k a_k x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = 0$$

Write out the first term of the first sum.

$$(2)(1)a_2 + \sum_{k=0}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} + \sum_{k=0}^{\infty} k a_k x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = 0$$

Combine the series.

$$2a_2 + \sum_{k=0}^{\infty} \left[(k+3)(k+2)a_{k+3} + ka_k + a_k\right] x^{k+1} = 0$$

a_2 and the quantity in square brackets must be zero.

$$(k+3)(k+2)a_{k+3} + (k+1)a_k = 0 \qquad a_2 = 0$$

Solve for a_{k+3}.

$$a_{k+3} = -\frac{k+1}{(k+3)(k+2)}a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$k=0: \quad a_3 = -\frac{0+1}{(0+3)(0+2)}a_0 = -\frac{1}{3\cdot 2}a_0$$

$$k=1: \quad a_4 = -\frac{1+1}{(1+3)(1+2)}a_1 = -\frac{2}{4\cdot 3}a_1$$

$$k=2: \quad a_5 = -\frac{2+1}{(2+3)(2+2)}a_2 = 0$$

$$k=3: \quad a_6 = -\frac{3+1}{(3+3)(3+2)}a_3 = -\frac{4}{6\cdot 5}\left(-\frac{1}{3\cdot 2}a_0\right) = (-1)^2\frac{4\cdot 1}{6\cdot 5\cdot 3\cdot 2}a_0$$

$$k=4: \quad a_7 = -\frac{4+1}{(4+3)(4+2)}a_4 = -\frac{5}{7\cdot 6}\left(-\frac{2}{4\cdot 3}a_1\right) = (-1)^2\frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}a_1$$

$$k=5: \quad a_8 = -\frac{5+1}{(5+3)(5+2)}a_5 = 0$$

$$\vdots$$

The general formula is

$$a_{3m} = (-1)^m\frac{[(3m-2)(3m-5)\cdots 4\cdot 1]^2}{(3m)!}a_0$$

$$a_{3m+1} = (-1)^m\frac{[(3m-1)(3m-4)\cdots 5\cdot 2]^2}{(3m+1)!}a_1$$

$$a_{3m+2} = 0.$$

Therefore, the general solution is

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} a_m x^m \\
&= a_0 + a_1 x + a_2 x^2 + \sum_{m=1}^{\infty} a_{3m} x^{3m} + \sum_{m=1}^{\infty} a_{3m+1} x^{3m+1} + \sum_{m=1}^{\infty} a_{3m+2} x^{3m+2} \\
&= a_0 + a_1 x + (0)x^2 + \sum_{m=1}^{\infty} (-1)^m \frac{[(3m-2)(3m-5)\cdots 4\cdot 1]^2}{(3m)!} a_0 x^{3m} \\
&\qquad + \sum_{m=1}^{\infty} (-1)^m \frac{[(3m-1)(3m-4)\cdots 5\cdot 2]^2}{(3m+1)!} a_1 x^{3m+1} + \sum_{m=1}^{\infty} (0) x^{3m+2} \\
&= a_0 + a_1 x + a_0 \sum_{m=1}^{\infty} (-1)^m \frac{[(3m-2)(3m-5)\cdots 4\cdot 1]^2}{(3m)!} x^{3m} \\
&\qquad + a_1 \sum_{m=1}^{\infty} (-1)^m \frac{[(3m-1)(3m-4)\cdots 5\cdot 2]^2}{(3m+1)!} x^{3m+1},
\end{aligned}$$

where a_0 and a_1 are arbitrary constants. Differentiate it with respect to x.

$$\begin{aligned}
y'(x) = a_1 + a_0 \sum_{m=1}^{\infty} (-1)^m \frac{[(3m-2)(3m-5)\cdots 4\cdot 1]^2}{(3m)!} (3m x^{3m-1}) \\
+ a_1 \sum_{m=1}^{\infty} (-1)^m \frac{[(3m-1)(3m-4)\cdots 5\cdot 2]^2}{(3m+1)!} (3m+1) x^{3m}
\end{aligned}$$

Apply the initial conditions to determine a_0 and a_1.

$$y(0) = a_0 = 0$$

$$y'(0) = a_1 = 1$$

Therefore,

$$y(x) = x + \sum_{m=1}^{\infty} (-1)^m \frac{[(3m-1)(3m-4)\cdots 5\cdot 2]^2}{(3m+1)!} x^{3m+1}.$$